Int. J. Solids Structures Vol. 21, No. 7, pp. 767–772, 1985
Printed in Great Britain.

0020–7683/85 $3.00 + .00

THE SINGULARITIES OF NONUNIFORMLY MOVING DISLOCATIONS

XANTHIPPI MARKENSCOFF
Department of Mechanical and Environmental Engineering, University of California, Santa Barbara, Santa Barbara, CA 93106, U.S.A.

Abstract—The singularities in the solution for the radiation from nonuniformly moving dislocations are analyzed. Singular asymptotics of integrals are essential for obtaining the limiting behavior on the slip-plane, as well as the field near the current position of the dislocation.

1. INTRODUCTION

In the course of our work on the nonuniform motion of dislocations during the past several years, we encountered several unexpected difficulties that are due to the type of the singularity of the dislocation. Here, we analyze and unify the origins of these singularities as they relate to singular asymptotics of integrals. For simplicity, we restrict attention to the screw dislocation and we omit extensive discussion of the edge and the loop since the nature of the singularities that they exhibit is exactly the same. We also restrict attention to isotropic materials.

2. THE FIELDS RADIATED FROM A NONUNIFORMLY MOVING DISLOCATION

This problem has been treated in [1]. A screw dislocation is considered at rest at the origin in a three-dimensional space, and at time $t = 0$ it starts moving on the $z = 0$ plane according to $x = l(t)$ or $t = \eta(x)$. Finding the fields radiated from the moving dislocation reduces to a problem in the two-dimensional space $z \geq 0$ where the displacement field $u_x \equiv 0$, $u_y = u(x, z, t)$, $u_z \equiv 0$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} = b^2 \frac{\partial^2 u}{\partial t^2}, \tag{1}$$

with boundary conditions:

$$u(x, 0, t) = \frac{\Delta u}{2} [H(x - l(t)) - H(x)] \quad \text{for} \quad t \geq 0. \tag{2}$$

The solution for a static dislocation at $x = 0$ must be superposed to the solution of this problem. The solution was sought by Laplace transform techniques in time ($t \rightarrow s$) and space ($x_1 \rightarrow s\lambda$), and it was computed for the strain component $\partial u/\partial z$, separately for $z \neq 0$ and for $z = 0$ (that is, on the slip-plane, where it is frequently of practical interest).

In order to obtain the solution in [1] on the slip-plane $z = 0$, we had to distinguish the field points (x, t) between those that the dislocation had gone through during its motion (region II in Fig. 1 in [1]) and those that had not (regions I, III in Fig. 1 in [1]).

After application of the transforms to (1) and (2) and setting $z = 0$ we obtain

$$\frac{\partial \hat{u}}{\partial z}(x, 0, s) = \frac{\Delta u}{2} \frac{s}{2\pi i} \int_{Br} \int_0^{\infty} (b^2 - \lambda^2) e^{s\lambda(x-\xi) - s\eta(\xi)} \, d\xi \, d\lambda. \tag{3}$$

For regions I and III we can interchange the order of integration in ξ and λ in (3) and obtain by the Cagniard-de Hoop scheme

$$\frac{\partial u}{\partial t}(x, 0, t) = -\frac{\Delta u}{2\pi}\int_0^{\infty}\frac{(t-\eta(\xi))H(t-\eta(\xi)-rb)\,\mathrm{d}\xi}{(x-\xi)^2\sqrt{(t-\eta(\xi))^2-r^2b^2}}, \tag{4}$$

where $r = |x - \xi|$ and the upper limit $\xi_*(x, z, t)$ of the integration is defined from the root of the argument of the step-function, i.e.

$$t - \eta(\xi_*) - \sqrt{(x-\xi_*)^2 + z^2}\,b = 0. \tag{5}$$

For subsonic motion, i.e. $\eta'(\xi) > b$, there is only one real root to (5), while for supersonic motion there may be several roots, which means that several disjoint intervals of the history of the motion may contribute to the solution at one field point. This also changes the nature of the solution and produces delta-function stress-singularities at the Mach wave-fronts[2].

The solution (4) is convergent of regions I and III since $\xi = x$ is not within the range of integration. However, for region II, where $\xi = x$, the interchange of orders of integration in λ, and ξ in (3), is not permissible since the integral

$$\int_{Br}(b^2-\lambda^2)^{1/2}e^{s\lambda(x-\xi)-s\eta(\xi)}\,\mathrm{d}\xi\,\mathrm{d}\lambda$$

does not converge absolutely.

In order to circumvent this difficulty we used a "trick" in [1], namely we subtracted the singularity from the integrand in (3) and added it. This way, the integrand contained no singularity and the interchange of the orders of integration was made valid, while the added term (which is the solution for a dislocation moving from rest with constant velocity equal to the instantaneous velocity $\dot{l}(t)$) could be integrated explicitly. So the solution for region II is:

$$\frac{\partial u}{\partial z}(x, 0, t) = -\frac{\Delta u}{2\pi}\int_0^{\infty}\left\{\frac{(t-\eta(\xi))H(t-\eta(\xi)-|x-\xi|b)}{\sqrt{(t-\eta(\xi))^2-(x-\xi)^2b^2}(x-\xi)^2}\right.$$
$$\left.-\frac{(t-\eta(x)-\eta'(x)(\xi-x))H(t-\eta(x)-\eta'(x)(\xi-x)-|x-\xi|b)}{\sqrt{(t-\eta(x)-\eta'(x)(\xi-x))^2-(x-\xi)^2b^2}(x-\xi)^2}\right\}\mathrm{d}\xi$$
$$+\frac{\Delta u}{2\pi}\frac{\operatorname{sgn} x}{t-\eta(x)}\sqrt{\left(\frac{t+\eta'(x)x-\eta(x)}{x}\right)^2-b^2}\,H(t-|x|b-\eta(x)+x\eta'(x))$$
$$-\frac{\Delta u}{2\pi}\frac{1}{x}.$$

By applying the transform again to (1) and (2) we obtained the solution for $z \neq 0$ without need to distinguish between different regions:

$$\frac{\partial u}{\partial z}(x, z, t) = -\frac{\Delta u}{2\pi}\int_0^{\infty}\frac{(t-\eta(\xi))(x-\xi)^2H(t-\eta(\xi)-rb)}{r^4[(t-\eta(\xi))^2-r^2b^2]^{1/2}}\,\mathrm{d}\xi$$
$$+\frac{\Delta u}{2\pi}z^2\frac{\partial}{\partial t}\int_0^{\infty}\frac{(t-\eta(\xi))^2H(t-\eta(\xi)-rb)}{r^4[(t-\eta(\xi))^2-r^2b^2]^{1/2}}\,\mathrm{d}\xi-\frac{\Delta u}{2\pi}\frac{x}{x^2+z^2}. \tag{6}$$

We may note that the order of integration and differentiation in the second integral in (5) cannot be interchanged because, if differentiated in t, the integrand will possess non-integral singularities at the upper limit of integration ξ_*.

3. SINGULAR ASYMPTOTICS OF INTEGRALS AND LIMITING BEHAVIOR AS $z \to 0$

Given the solution for $z = 0$ and $z \neq 0$, one would like to be able to obtain the former in the limit of the latter as $z \to 0$. However, if one tried to expand the integrand in powers of z, the coefficients would be integrals in ξ that would be increasingly divergent (at $x = \xi$). Thus regular asymptotic (Taylor) expansion of integrals do not work and we need to consider singular asymptotics of integrals for that purpose. Singular asymptotic series expansions of integrals have been developed by Callias[3] and Callias and Markenscoff[4]. For fairly general functions f, the series:

$$\int_0^\infty f(y, x)\,dx = \sum_{m=0}^{\infty} a_m \epsilon^m + \sum_{m=1}^{\infty} b_m \epsilon^m \ln \epsilon, \tag{7}$$

where $y \equiv \epsilon/x$ is an asymptotic series with the coefficients a_m and b_m given by [4]:

$$a_0 = \int_0^\infty f(0, x)\,dx$$

$$a_m = -\frac{1}{m!(m-1)!}\int_0^\infty \ln x \frac{\partial^{2m} f(0, x)}{\partial y^m \partial x^m}\,dx + C_m \frac{1}{m!(m-1)!} + \frac{\partial^{2m-1} f(0, 0)}{\partial y^m \partial x^{m-1}} + \mathcal{L}_m(\phi)$$

$$\left[\text{where } \mathcal{L}_m(\phi) = \frac{1}{(m-1)!}\int_0^\infty d\xi \ln \xi \frac{\partial}{\partial \xi} \xi^m R_{m+1}\left(\frac{1}{\xi}\right),\right.$$

$$\left. R_{m+1}(y) = \phi(y) - \sum_{k=0}^{m} \frac{1}{k!}\phi^{(k)}(0) y^k \right]$$

$$b_m = \frac{-1}{m!(m-1)!}\frac{\partial^{2m-1} f(0, 0)}{\partial y^m \partial x^{m-1}}$$

where $C_m = \sum_1^{m-1} 1/n$.

For more specific forms of f, i.e. $f(\epsilon/x, x) = f(\epsilon/x), \phi(x)$ the coefficients a_m and b_m have been obtained by Callias[3].

The above series has the feature that it contains, besides integral powers of ϵ, an infinity of logarithmic terms in ϵ. We can easily see that the integrals in (6) are of the form $\int f(\epsilon/x)\phi\,dx$, if we write

$$\int_0^{\xi_*} \frac{(t - \eta(\xi))(x - \xi)^2\,d\xi}{[(x - \xi)^2 + z^2]^2 \sqrt{(t - \eta(\xi))^2 - r^2 b^2}}$$

$$= \frac{1}{z^2}\frac{z^2 (x - \xi)^2 (t - \eta(\xi))\,d\xi}{[(x - \xi)^2 + z^2]^2 \sqrt{(t - \eta(\xi))^2 - r^2 b^2}}$$

$$= \frac{1}{z^2}\int_0^{\xi_*} \frac{z^2/(x - \xi)^2 (t - \eta(\xi))\,d\xi}{[1 + z^2/(x - \xi)^2]^2 \sqrt{(t - \eta(\xi))^2 - r^2 b^2}}$$

$$= \frac{1}{z^2}\int_0^{\xi_*} f\left(\frac{z}{x - \xi}\right)\phi(\xi, z)\,d\xi. \tag{9}$$

Now, in order to make the series (7) directly applicable we extend the upper limits of integration in (9) to infinity by writing as in [4]:

$$\frac{1}{z^2}\int_0^{\xi_*} f\left(\frac{z}{x-\xi}\right)\phi(\xi, z)\, d\xi$$
$$\equiv \frac{1}{z^2}\int_0^{\xi_*} f\left(\frac{z}{x-\xi}\right)\phi(\xi, z)\chi(\xi)\, d\xi + \frac{1}{z^2}\int_0^{\xi_*} f\left(\frac{z}{x-\xi}\right)\phi(\xi, z)(1-\chi(\xi))\, d\xi, \quad (10)$$

where $\chi(\xi) \in C_0^\infty(-\infty, \infty)$, and is such that its support is contained in $(-\xi_*, \xi_*)$ and $\chi(\xi) = 1$ for $-a \leq x \leq a$, $0 < a < \xi_0$. The function $\chi(\xi)$ is otherwise arbitrary and the final result should be independent of it.

The first term on the right-hand side of (10) will be expanded asymptotically in z by using the series (7), while the second term, which contains no singularities at $\xi = x$, since $1 - \chi(\xi) = 0$ in the neighborhood of x, admits regular series expansion, and the limit as $z \to 0$ can be taken straight-forwardly, so that the integral converges to:

$$\int_0^{\xi_*(x,0,t)} \frac{(t-\eta(\xi))(1-\chi(\xi))\, d\xi}{(x-\xi)^2\sqrt{(t-\eta(\xi))^2-(x-\xi)^2 b^2}}. \quad (11)$$

Let us consider now the first term as the R.H.S. of (10):

$$I_1 \equiv \frac{1}{z^2}\int_0^{\xi_*} \frac{z^2(x-\xi)^2(t-\eta(\xi))\chi(\xi)\, d\xi}{[1+z^2/(x-\xi)^2]^2\sqrt{(t-\eta(\xi))^2-(x-\xi)^2 b^2}}$$
$$\equiv \frac{1}{z^2}\int_{-\infty}^{+\infty} f(y)\phi(\xi', z)\chi(\xi')\, d\xi',$$

where

$$y \equiv \frac{z}{x-\xi}, \quad \xi' = x - \xi, \quad f(y) = \frac{y^2}{(1+y^2)^2}$$

and

$$\phi(\xi, x) = \frac{(t-\eta(\xi))\chi(\xi)}{\sqrt{(t-\eta(\xi))^2-(x-\xi)^2b^2-z^2b^2}}$$

We can expand $\phi(\xi', z) = \sum_{k=0}^{\infty} 1/k!\partial^k\phi(\xi, 0)z^k$ and consider the expansion (to order z^2 only for the limit $z \to 0$)

$$I_1 \equiv \frac{1}{z^2}\int_{-\infty}^{\infty} f\left(\frac{z^2}{\xi'^2}\right)\phi(\xi', z)\, d\xi' \sim \sum_0^2 z^m \sum_{j=0}^{m} A_{m-j,j} \quad (12)$$

where $A_{m-j,j}$ are found from (8) by adding two series (for the integrals from $-\infty$ to 0, and from 0 to ∞):

$$A_{0,0} = A_{1,0} = A_{1,1} = A_{2,0} = 0$$
$$A_{0,1} = \frac{(t-\eta(x))\chi(x)}{[(t-\eta(x))^2]^{1/2}} = \frac{\pi}{2}$$
$$A_{0,2} = \frac{(-1)}{1!}\int_{-\infty}^{\infty} \ln|\xi - x|\left(\frac{\partial}{\partial\xi}\right)^2\left[\frac{(t-\eta(\xi))\chi(\xi)}{\sqrt{(t-\eta(\xi))^2-b^2(x\xi)^2}}\right] d\xi.$$

Thus

$$I_1 \sim \frac{\pi}{2z} - \int_{-\infty}^{\infty} \ln|\xi - x|\left(\frac{\partial}{\partial\xi}\right)^2\left[\frac{(t-\eta(\xi))\chi(\xi)}{\sqrt{(t-\eta(\xi))^2-b^2(x-\xi)^2}}\right] d\xi + \dots \quad (13)$$

We next consider the second integral in (6) for which we write

$$I_2 = \frac{1}{z^2}\int_{-\infty}^{\infty}\left(\frac{z}{\xi - x}\right)^4 \frac{1}{\left(1 + \left(\frac{z}{\xi - x}\right)^2\right)^2} \frac{2(t - \eta(\xi))\chi(\xi)\, d\xi}{\sqrt{(t - \eta(\xi))^2 - (x - \xi)^2 b^2 - z^2 b}}$$
$$+ \frac{1}{z^2}\int_{-\infty}^{\infty}\left(\frac{z}{\xi - x}\right)^4 \frac{1}{\left(1 + \left(\frac{z}{\xi - x}\right)^2\right)} \frac{(t - \eta(\xi))^3\chi(\xi)\, d\xi}{[(t - \eta(\xi))^2 - (x - \xi)^2 b^2 - z^2 b^2]^{3/2}}$$
$$+ z^2 \frac{\partial}{\partial t}\int_0^{\xi_*} \frac{(t\eta(\xi))^2(1 - \chi(\xi))\, d\xi}{[(x - \xi)^2 + z^2]^2\sqrt{(t - \eta(\xi))^2 - (x - \xi)^2 b^2 - z^2 b^2}} . \quad (14)$$

The last integral in eqn (14) is zero near $x = \xi$ and converges to zero as $z \to 0$ elsewhere. For the first two integrals in eqn (14), the coefficients in the asymptotic series (12) are found, respectively, to be: For the integral

$$I_{21} \equiv \frac{1}{z^2}\int_{-\infty}^{\infty}\left(\frac{z}{\xi - x}\right)^4 \frac{1}{(1 + (z/(\xi - x))^2)^2} \frac{2(t - \eta(\xi))\chi(\xi)\, d\xi}{\sqrt{(t - \eta(\xi))^2 - (x - \xi)^2 b^2 - z^2 b^2}} ,$$

$$A_{0,0} = A_{1,0} = A_{1,1} = A_{2,0} = A_{0,2} = 0$$
$$A_{0,1} = \frac{(-\frac{1}{2})(-1)\pi}{2!} \frac{(2(t - \eta)(x))\chi(x)}{\sqrt{(t - \eta(x))^2}} = \pi \quad (15)$$

so that $I_{21} \sim \pi/z$, and for the integral

$$I_{22} \equiv \frac{1}{z^2}\int_{-\infty}^{\infty}\left(\frac{z}{x - \xi}\right)^2 \frac{1}{\left(1 + \left(\frac{z}{x - \xi}\right)^2\right)^2} \frac{(t - \eta(\xi))^3\chi(\xi)\, d\xi}{[(t - \eta(\xi))^2 - (x - \xi)^2 b^2 - z^2 b^2]^{3/2}} ,$$

$$A_{0,0} = A_{1,0} = A_{1,1} = A_{2,0} = A_{0,2} = 0$$
$$A_{0,1} = \frac{(-\frac{1}{2})(-1)\pi}{1} \frac{(-1)(t - \eta(x))^3\chi(x)}{[(t - \eta(x))^2]^{3/2}} = -\frac{\pi}{2} \quad (16)$$

so that

$$I_{22} \sim -\frac{\pi}{2z} .$$

Considering the sum $-\Delta u/2\pi(I_1 - I_{21} - I_{22})$ and eqn (11), we obtain for the strain $\partial u/\partial z(x, z, t)$ the limit as $z \to 0$

$$\frac{\partial u}{\partial z}(x, 0, t0) = \frac{\Delta u}{2\pi}\int_{-\infty}^{\infty} \ln|\xi - x|\left(\frac{\partial}{\partial \xi}\right)^2\left[\frac{(t - \eta(\xi))\chi(\xi)}{\sqrt{(t - \eta(\xi))^2 - b^2(x - \xi)^2}}\right] d\xi$$
$$- \int_0^{\xi^*} \frac{(t - \eta(\xi))(1 - \chi(\xi))\, d\xi}{(\xi - x)^2\sqrt{(t - \eta(\xi))^2 - b^2(x - \xi)^2}} . \quad (17)$$

We have thus seen that although the stress is not singular at field points on the slip-plane in region II (i.e. the region that the dislocation has swept during its motion), to find the stress at these points (in the limit as $z \to 0$) requires singular asymptotic expansions. If one tried to obtain the solution for these points by using the Green's function approach, one would still encounter the same difficulties and could also have to use singular asymptotics. The same is true also for dislocation loops: that is, if one wishes to find the solution in the limit ($z \to 0$) as a point on the plane of the loop is approached, and this point has been previously swept by the loop, then singular asymptotics of integrals are required. For the expression of the solution of arbitrarily expanding loops, and its calculation for some special motions, the reader is referred to[5, 6]. For an edge dislocation longitudinal and equivoluminal waves are coupled but the solution for general dislocation motion may still be expressed[7] in integral form analogous to eqn (6), so that the corresponding singularities are identical.

4. NEAR-FIELD OF A NONUNIFORMLY MOVING DISLOCATION

The dislocation, modelled in classical elasticity theory as a stlep-function discontinuity in the displacement, possesses an ϵ^{-1} singularity near the core. When the dislocation moves in a steady-state motion, it still has the same type of singularity, whereas for an accelerating motion it also exhibits an $\ln \epsilon$ singularity near the core[4]. The coefficient of the logarithmic singularity for a screw dislocation moving according to $x = l(t)$ is[4]

$$\frac{\Delta u}{4\pi} b^2 \frac{\ddot{l}(t)}{\left(1 - \frac{\dot{l}(t)^2}{c_2^2}\right)} .$$

That is, it depends on the instantaneous value of the acceleration $\ddot{l}(t)$. Eshelby[8] has obtained a logarithmic singularity for the case of constant acceleration when the integrals in eqn (6) are elliptic. However, for general motion $l(t)$, the solution (6) involves integrals of general functions, for which asymptotic series–as in eqn (7)–are required.

In order to obtain[4] the core singularity form (6) we expanded the integrals around the current position of the dislocation, i.e. for field points $x = l(t) + \epsilon X(\theta)$, $z = \epsilon Z(\theta)$. In this case, the integrals diverge at the upper limit of integration ξ_*, which may become zero by change of variable $\xi' = \xi_* - \xi$, so that the integrals in eqn (6) are of the form

$$\frac{1}{\xi'^2} \int_0^{\xi^*(\epsilon)} f\left(\frac{\epsilon}{\xi'}, \xi', \epsilon\right) d\xi',$$

and the expansion is obtained by application of the series (7).

We have thus seen that singular asymptotics of integrals are at the root of the solution of nonuniformly moving dislocations (screw, edge, or loops) even for points where the solution itself is not singular.

Acknowledgment—The author acknowledges the contributions of Costas Callias to Part III.

REFERENCES

1. X. Markenscoff, The transient motion of a nonuniformly moving dislocation, *J. Elasticity*, **10**, 193–201 (1980).
2. C. Callias and X. Markenscoff, The nonuniform motion of a supersonic dislocation, *Quart. Appl. Math.*, **38**, 323–330 (1980).
3. C. Callias, The heat equation with singular coefficients, *Commun. Math. Phys.*, **88**, 357–385 (1983).
4. C. Callias and X. Markenscoff, Singular asymptotics of integrals and radiation from nonuniformly moving dislocations, *Arch. Rat. Mech. Anal.* to appear.
5. T. Mura, Methods of continuously distributed dislocations. In *Mathematical Theory of Dislocations* (Edited by T. Mura), 25–48. ASME (1969).
6. X. Markenscoff and L. Ni, On the effect of dislocation loop curvature on elastic precursor decay, *J. Appl. Mech.* **51**, 753–758 (1984).
7. X. Markenscoff and R. J. Clifton, The nonuniformly moving edge dislocation, *J. Mech. Phys. Sol.*, **29**, 252–262 (1981).
8. J. D. Eshelby, The equation of motion of a dislocation, *Phys. Rev.*, **90**, 248–255 (1953).